IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 2

Tuesday 25 May 2004 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Write the make and model of your calculator in the appropriate box on your cover sheet *e.g.* Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

1. *[Maximum mark: 21]*

(i) Let $\sum_1^\infty u_n$ and $\sum_1^\infty v_n$ be two convergent series of positive terms.

(a) Show that $u_n v_n < v_n$ for large values of n. *[2 marks]*

(b) Show that $\sum_1^\infty u_n v_n$ converges, and hence show that $\sum_1^\infty u_n^2$ converges. *[2 marks]*

(ii) (a) Use the power series expansion of $\frac{1}{1+x}$ to find an expansion for $h(x) = \ln(1+x)$. For what values of x is your expansion convergent? *[5 marks]*

(b) Hence find the first three non-zero terms of the expansion of $g(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. *[4 marks]*

In parts (c), (d) and (e), give your answers correct to six decimal places.

(c) Use Simpson's rule with 10 intervals to estimate $\int_0^1 \frac{\mathrm{d}x}{1+x}$. *[2 marks]*

(d) Given that the fourth derivative of $\frac{1}{1+x}$ is $\frac{24}{(1+x)^5}$, find the maximum possible error in your approximation. *[2 marks]*

(e) Consider the function $f(x) = \frac{1}{1+x} - x$. Use a fixed point iteration to find the positive zero of this function. *[4 marks]*

2. *[Maximum mark: 21]*

(i) Let x, p, q and r be elements of a group with identity element e.

Solve for x

(a) the equation $pxq = r$; *[2 marks]*

(b) the simultaneous equations $\begin{cases} px^2 = q \\ x^3 = e \end{cases}$. *[4 marks]*

(ii) Let G be a finite group such that for every $x \in G, x^2 = e$, where e is the identity element.

(a) Show that xy and yx are inverses of each other, and deduce that G is commutative. *[4 marks]*

(b) Let H be a subgroup of G. Let a be an element of G not in H. The set aH is defined by

$$aH = \{ah \mid h \in H\}.$$

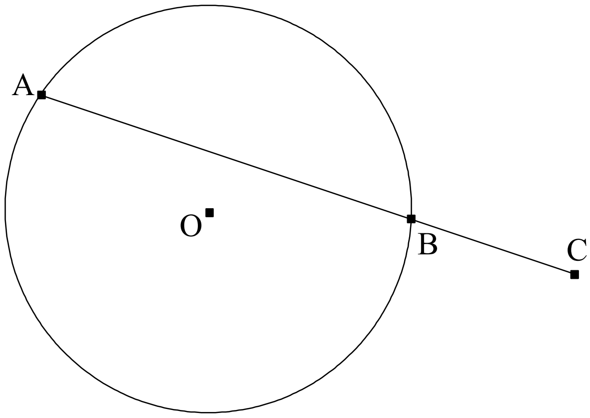
(i) Show that $H \cap aH = \varnothing$.

(ii) Show that $H \cup aH$ is a subgroup of G.

(iii) Show that the number of elements of $(H \cup aH)$ is twice the number of elements of H. *[11 marks]*

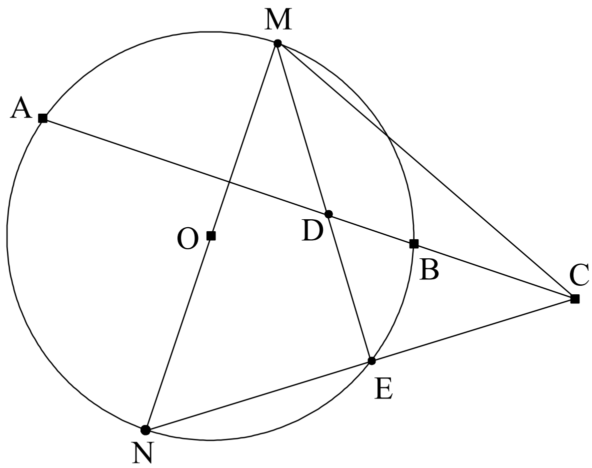
3. *[Maximum mark: 18]*

(i) The diagram below shows a fixed straight line [AC] and B is a fixed point between A and C. A variable circle with centre O passes through A and B.



(a) Find the locus of O. *[2 marks]*

In the next diagram, diameter [MN] is perpendicular to (AB). (NC) intersects the circle at E. (EM) intersects (AB) at D.



(b) Prove that D is a fixed point. *[6 marks]*

(c) Find the locus of E. *[3 marks]*

(ii) Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b$.

The tangent at the point $P(x_0, y_0)$ intersects the major axis at point M. The normal at P intersects the major axis at N.

(a) Find the x-coordinate of M and of N. *[5 marks]*

(b) Hence show that G, F, N, and M form a harmonic division, where G and F are the foci of the hyperbola. *[2 marks]*

4. *[Maximum mark: 18]*

(i) Find the first three positive integers satisfying the modular equation

$$2x \equiv 7 \pmod{17}$$ *[4 marks]*

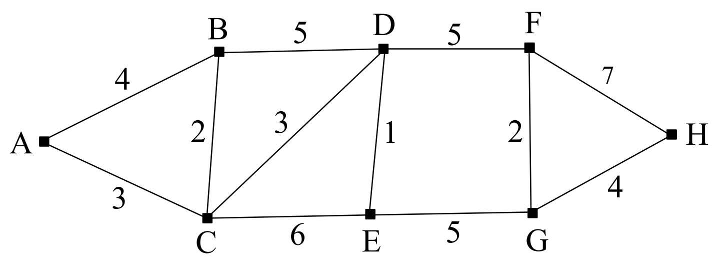
(ii) Consider the two recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1},\ b_n = a_{n-1} + 2b_{n-1}, \text{ with } a_0 = 1 \text{ and } b_0 = 2.$$

(a) By eliminating b_n and b_{n-1}, show that $a_n = 5a_{n-1} - 4a_{n-2}$. *[2 marks]*

(b) Find a_n and b_n. *[8 marks]*

(iii) Use Dijkstra's algorithm to find the shortest path from A to H in the following weighted graph. *[4 marks]*



5. *[Maximum mark: 22]*

(i) Resistors used in the construction of aircraft guidance systems have lifetimes that are normally distributed with a mean of 14000 hours and a standard deviation of 3000 hours.

(a) One resistor is taken at random from a large consignment. Find the probability that its lifetime is more than 12000 hours. *[2 marks]*

(b) An aircraft uses a system of five resistors, each working independently. The system is set up so that only one resistor is working at any given time. When that resistor stops working, another one starts working, until all five have failed.

(i) Find the probability that the lifetime of this system is more than 60000 hours. *[3 marks]*

(ii) Find the probability that at least two of the resistors last less than 12000 hours. *[5 marks]*

(iii) Find the probability that the average lifetime of these resistors is more than 13000 hours. *[3 marks]*

(ii) Cellular phones are designed to work for long periods of time before their batteries run down. A communications service provider uses phones from different suppliers. It is concerned about the variation among phone types in the number of hours that the batteries last in stand-by mode. A thousand phones were tested and the results are given in the table below.

Phone type	A	B	C	D	E
Hours in stand-by mode					
$x \ge 180$	88	148	158	144	62
$150 \le x < 180$	28	50	54	48	20
$120 \le x < 150$	30	40	40	46	18
$x < 120$	6	10	10	0	0

Test at the 5 % level of significance whether there is a connection between phone types and the number of hours that batteries last in stand-by mode. *[9 marks]*